1.6: Substitution Method and Exact Equations

As is usual with any substitution method, we wish to substitute in a new variable $v = \alpha(x, y)$ into the differential equation $dy/dx = f(x, y)$ so that the new differential equation $dv/dx = g(x, y)$ is one that we know how to solve.

Example 1. Using substitution, solve the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2.$$

Let $v = x+y+3$

$y = v - x - 3$

$$\frac{dy}{dx} = \frac{\partial y}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial y}{\partial v}\cdot\frac{dv}{dx} = -1 + 1\cdot\frac{dv}{dx}$$

So $-1 + \frac{dv}{dx} = v^2$

or $\frac{dv}{dx} = 1 + v^2$

Separable so

$x = \tan^{-1} v + C$

Plug in $v = x+y+3$ to solve for

$y = \tan(x-C) - x - 3.$

Definition 1. A **homogeneous** first-order differential equation is one that can be written in the form $\frac{dy}{dx} = f(\frac{y}{x})$. In this case we use the substitution $v = \frac{y}{x}$ so that

$$y = vx, \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Example 2. Solve the differential equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2. \tag{1}$$

Divide by x^2. $2\left(\frac{y}{x}\right)\frac{dy}{dx} = 4 + 3\left(\frac{y}{x}\right)^2$. Let $v = \frac{y}{x}$ so that

$y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then we get $2v\left(v + x\frac{dv}{dx}\right) = 4 + 3v^2$

or $x\frac{dv}{dx} = \frac{v^2+4}{2v}$ or $\frac{2v}{v^2+4}dv = \frac{dx}{x}$.

Thus $\ln(v^2+4) = \ln|x| + C$

or $v^2 + 4 = C|x| \implies y = \pm\sqrt{Cx^3 - 4x^2}$

Exercise 1. Solve the initial value problem

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad y(x_0) = 0 \quad (x_0 > 0).$$

Divide by x.

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2-y^2}}{x} = \frac{y}{x} + \sqrt{1-\frac{y^2}{x^2}}$$

$$v + x\frac{dv}{dx} = v + \sqrt{1-v^2} \quad \left(v = \frac{y}{x}\right)$$

Separable so

$$\sin^{-1} v = \ln x + C$$

Solve for $C = -\ln x_0$

Then $y = x\sin\left(\ln\frac{x}{x_0}\right)$.

Definition 2. A first-order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{2}$$

is called a **Bernoulli** equation. Using the substitution $v = y^{1-n}$ (2) becomes

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

which we can solve using the methods from the previous section.

Example 3. Rewriting ~~the~~ (1) in the form

$$\frac{dy}{dx} - \frac{3}{2x}y = \frac{2x}{y}$$

solve using the method described above.

$\frac{dy}{dx} + P(x)y = Q(x)y^n$ where $P(x) = \frac{-3}{2x}$, $Q(x) = 2x$ and $n = -$

Let $v = y^{1-n} = y^2$. Then the eqn becomes

$\frac{dv}{dx} - \frac{3}{x}v = 4x$. So $\rho(x) = e^{-\int 3/x\,dx} = e^{-3\ln x} = x^{-3}$.

Multiply and integrate as in Sect. 1.5 to get

$$x^{-3}v = \int x^{-3}\cdot 4x\,dx = -\frac{4}{x} + C$$

Thus $v = -4x^2 + Cx^3$ and $y = \pm\sqrt{Cx^3 - 4x^2}$

Exercise 2. Use the method of Bernoulli equations to solve

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}.$$

$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}$$

$v = y^{-1/3}$, so that

$$\frac{dv}{dx} - \frac{2}{x}v = -1$$

Then $\rho(x) = e^{-\int 2/x\,dx} = x^{-2}$.

and $x^{-2}v = \frac{1}{x} + C$

Solve for y to get $y = \dfrac{1}{(x+Cx^2)^3}$.

Example 4.

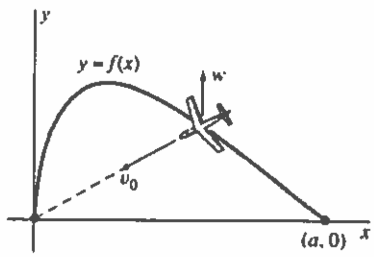

FIGURE 1.6.4. The airplane headed for the origin.

An airplane departs from point $(a, 0)$ located due East of its intended destination at the origin. The wind is blowing due North with a constant speed of w. We assume the plane is alway pointed at its destination so that its velocity vector v_0 is also. (Figure 1.6.4)

Hence the trajectory $y = f(x)$ of the plane satisfies

$$\frac{dy}{dx} = \frac{1}{v_0 x}\left(v_0 y - w\sqrt{x^2+y^2}\right). \qquad (3)$$

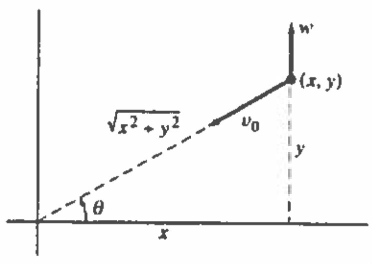

FIGURE 1.6.5. The components of the velocity vector of the airplane.

Let $k = \frac{w}{v_0}$ and the substitution $y = xv$ to show that (3) has the solution

$$y(x) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right]. \qquad (4)$$

$$\frac{dy}{dx} = \frac{y}{x} - k\left[1 + \left(\frac{y}{x}\right)^2\right]^{1/2}$$

$v = \frac{y}{x}$, so $y = vx$ and $y' = v + xv'$

$$\int \frac{dv}{\sqrt{1+v^2}} = -\int \frac{k}{x}\,dx$$ ~~an~~

$$\ln\left(v + \sqrt{1+v^2}\right) = -k\ln x + C$$

But $v(a) = y(a)/a = 0$ so $C = k\ln a$.

Thus $v = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right]$

and $y = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right]$

Exercise 3. Using (4), find the maximum amount by which the plane is blown of course when a=200 mi, v_0=500 mi/h, and w= 100 mi/h.

From (4), $y(x) = 100\left[\left(\frac{x}{200}\right)^{4/5} - \left(\frac{x}{200}\right)^{6/5}\right]$

Then $\frac{dy}{dx} = \frac{1}{2}\left[\frac{4}{5}\left(\frac{x}{200}\right)^{-1/5} - \frac{6}{5}\left(\frac{x}{200}\right)^{1/5}\right]$

which is 0 when $\left(\frac{x}{200}\right)^{2/5} = \frac{2}{3}$

Plugging this in we get $y_{max} \approx 14.81$ miles north.

Definition 3. An **exact** differential equation is of the form

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \quad \text{or} \quad M(x,y)dx + N(x,y)dy = 0, \tag{5}$$

where $F(x, y)$ is a differentiable function of x and y. Recall that if F is twice differentiable then a necessary condition is that

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x}.$$

Example 5. Find the general solution to the differential equation

$$y^3dx + 3xy^2dy = 0.$$

$\int y^3 dx = xy^3 + C_1$

$\int 3xy^2 dy = xy^3 + C_2.$

$\Rightarrow$ This is exact and

$xy^3 = C$ or $y = \left(\frac{C}{x}\right)^{1/3}.$

Theorem 1. **(Criterion for Exactness)** Suppose that the functions $M(x, y)$ and $N(x, y)$ are continuous and have continuous partial derivatives on the open rectangle R. Then the differential equation (5) is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Homework. Example 9 from the book. Read the Section and Examples on Reducible Second-Order Equations. 1-21, 31-39, 43-51, 57-61 (odd)